PLANE SECTIONS OF CENTRALLY SYMMETRIC CONVEX BODIES

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ABSTRACT

The note contains an example of three plane convex centrally symmetric figures P_1, P_2, P_3 such that no centrally symmetric 3-dimensional body has three coaxial central affinely equivalent to P_1, P_2, P_3 respectively.

In 1933 S. Banach and S. Mazur [1] proved that the space C[0, 1] of all continuous functions on the segment [0, 1] is universal, with respect to isometry, for all separable Banach spaces. This means that for each separable Banach space X there exists a subspace of $C[0, 1]$ which is isometric to X.

Simultaneously the following question arose. Does there exist a finite dimensional space universal (with respect to isometry) for all two-dimensional Banach spaces? in geometric language this means: Is there an n-dimensional, centrally symmetric, convex body K such that for each plane centrally symmetric convex set P we can find a two-dimensional section $\tilde{P}$ through the center of K, such that $\tilde{P}$ is affinely equivalent to P.

The answer is negative. B. Grünbaum [3] established that there is no 3-dimensional K with this property, while C. Bessaga [2] proved the non-existence for general n. Additional results were obtained by V. Klee [4].

It follows from these proofs that there exists a number i_n, and plane, centrally symmetric convex sets $P_1, \dots, P_{i_n}$, with the property: no n-dimensional centrally symmetric convex body K has two-dimensional sections $\tilde{P}_1, \dots, \tilde{P}_{i_n}$, through its center, such that $\tilde{P}_j$ is affinely equivalent to P_j for $j = 1, \dots, i_n$.

Basing on Bessaga's arguments an estimate of i_n could be obtained; however, it would probably be very far from the minimal possible value of i_n.

It may be conjectured that $\inf i_n = n + 1$, but this conjecture is still unsolved even for $n = 3$.

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In this note we shall consider a related problem of A. Pełczyński.

Suppose that we consider not all sections of a 3-dimensional centrally symmetric convex body K, but only sections which contain some fixed straight line passing through the center of K. What is the least number k of plane, centrally symmetric convex sets $P_1, .., P_k$, with the property: For no centrally symmetric 3-dimensional convex body K does there exist a straight line L through the center of K, and two-dimensional sections $\tilde{P}_1, .., \tilde{P}_k$ through L, such that $\tilde{P}_j$ is affinely equivalent to P_j for $j = 1, .., k$.

Clearly, $k \geqq 3$. In the present note we shall show that $k = 3$.

Let P_1^ε be a square, P_2^ε a circle, and P_3^ε a square of side 2 with corners rounded off by circular arcs of radius ε (see Figure 1.). Clearly, only P_3^ε depends on ε.

THEOREM. *There exists an $\varepsilon > 0$ such that there exists no centrally symmetric, 3-dimensional convex body K_ε admitting a line L through its center and sections $\tilde{P}_1^\varepsilon$, $\tilde{P}_2^\varepsilon$, $\tilde{P}_3^\varepsilon$ through L, such that $\tilde{P}_j^\varepsilon$ is affinely equivalent to P_j^ε; $j = 1, 2, 3$.*

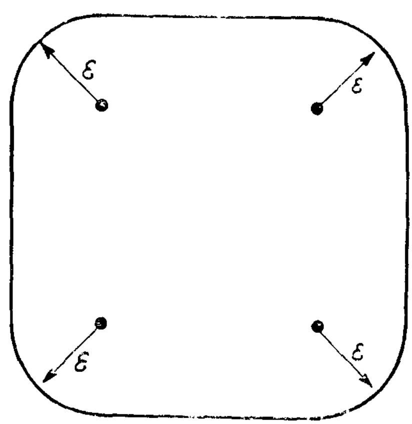

Fig. 1.

Proof. Suppose that for each $\varepsilon > 0$ such body K_ε exists. Let t be one of the intersection points of L with the boundary B_ε of K_ε. Then t can not be an exposed point of $\tilde{P}_1^\varepsilon$ because t belongs also to the sections $\tilde{P}_2^\varepsilon$ and $\tilde{P}_3^\varepsilon$, which are smooth. (See [3]). Hence t is not an extermal point of $\tilde{P}_1^\varepsilon$. Since $\tilde{P}_2^\varepsilon$ is strictly convex, t must belong to the relative interior of one of the curved arcs of $\tilde{P}_3^\varepsilon$. (See [3].

Being interested only in affine properties we may assume, without loss of generality, that $\tilde{P}_2^\varepsilon$ is a unit circle in the plane $z = 0$, and that $\tilde{P}_3^\varepsilon$ is homothetic to P_3^ε and situated in the plane $y = 0$. Then $\tilde{P}_1^\varepsilon$ is a parallelogram in some plane $z = \alpha y$. We shall investigate the relationship between α and ε.

Let p_i be the boundary of $\tilde{P}_j^\varepsilon, j = 1, 2, 3$. The three curves p_1, p_2, p_3 intersect at the point t. Since p_3 is a normal section of B_ε, the normal curvature κ_3 of B_ε in direction p_3 (at t) is equal to the total curvature of p_3, which is $\geqq 1/\varepsilon$. The total curvature of p_2 at t is equal to 1, hence the normal curvature κ_2 of p_2 is at most 1.

Now B_ε is convex, and p_1 is a straight line in a neighborhood of t; hence the minimal curvature of B_ε at t is $\kappa_1 = 0$. The maximal normal curvature κ_2^0 of B_ε at t is in a direction perpendicular to that of p_1. Using Euler's formula we may express the normal curvatures of B_ε in directions p_3 and p_2 by

$$\kappa_3 = \kappa_2^0 \cos^2 \beta$$

$$\kappa_2 = \kappa_2^0 \sin^2 \beta,$$

where β is the angle between P_2 and P_1 at the point t. Therefore $\kappa_3/\kappa_2 = \mathrm{ctg}^2 \beta$; since $\kappa_3/\kappa_2 \geqq 1/\varepsilon$ it follows that $\beta \to 0$ for $\varepsilon \to 0$, which trivially implies that $\alpha \to 0$ when $\varepsilon \to 0$.

Since for each ε the points $(1, 0, 0)$ and $(-1, 0, 0)$ belong to B_ε it follows that for $\varepsilon \to 0$ the set $\tilde{P}_1^\varepsilon = K_\varepsilon \cap \{(x, y, z) : z = \alpha y\}$ tends to

$$K_\varepsilon \cap \{(x, y, z) : z = 0\} = \{(x, y, z) : z = 0, \alpha^2 + y^2 \leqq 1\}.$$

But this is impossible because $\tilde{P}_2^\varepsilon$ is a unit circle and $\tilde{P}_1^\varepsilon$ is a parallelogram.

This completes the proof of the theorem.

Using standard approximation and compactness arguments it is easy to deduce from the theorem the following

COROLLARY. *There exist centrally symmetric convex polygons P_1, P_2, P_3, such that no centrally symmetric 3-dimensional convex body has three coaxial central sections affinely equivalent to P_1, P_2, respectively P_3. Moreover, P_1 may be chosen as a square, and p_2 as a regular polygon of sufficiently many sides.*

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